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## Rotating cylindrical wormholes and energy conditions

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We seek wormholes among rotating cylindrically symmetric configurations in general relativity. Exact wormhole solutions are presented with such sources of gravity as a massless scalar field, a cosmological constant, and a scalar field with an exponential potential. However, none of these solutions are asymptotically flat, which excludes the existence of wormhole entrances as local objects in our Universe. To overcome this difficulty, we try to build configurations with flat asymptotic regions using the cut-and-paste procedure: on both sides of the throat, a wormhole solution is matched to a properly chosen region of flat space-time at some surfaces  $\Sigma_-$  and  $\Sigma_+$ . It is shown, however, that if the source of gravity in the throat region is a scalar field with an arbitrary potential, then one or both thin shells appearing on  $\Sigma_-$  and  $\Sigma_+$  inevitably violate the null energy condition. Thus, although rotating wormhole solutions are easily found without exotic matter, such matter is still necessary for obtaining asymptotic flatness.

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### 1. Introduction

Wormholes are hypothetical “bridges” or “tunnels” which connect different universes or different large or infinite regions of the same space-time. They are a subject of active discussion since their possible existence can lead to physical effects of great interest, such as realizable time machines or shortcuts between distant parts of the Universe, in particular, across black hole horizons.<sup>1–3</sup> Unusual observable effects can be predicted under the assumption that wormholes can exist on astrophysical scales of times and distances.<sup>4–6</sup>

As is well known, the existence of a static wormhole geometry in the framework of general relativity requires the presence of “exotic”, or phantom matter, that is,

matter violating the weak and null energy condition (WEC and NEC), at least in a certain neighborhood of the throat,<sup>1-3,7</sup> the narrowest place in a wormhole. This conclusion, however, rests on the assumption that the throat is a compact 2D surface, having a finite (minimum) area.<sup>7</sup> In other words, a wormhole entrance looks from outside as a local object like a star or a black hole.

Since macroscopic exotic matter has not been observed in laboratory or in the Universe (except for the possible phantom dark energy), it is natural to try to obtain phantom-free (i.e., without matter violating the NEC and WEC) wormholes by abandoning some of the assumptions of the Hochberg-Visser (HV) theorem.<sup>7</sup> One of the simplest ways is to consider cylindrical symmetry thus rejecting the compact nature of the throats. Then, instead of starlike objects, we deal with objects like cosmic strings, infinitely extended along a certain direction. One can also consider nonstatic, rotating configurations, which can again be done in the framework of cylindrical symmetry.

Examples of phantom-free wormholes are known in some extensions of general relativity, such as the Einstein-Cartan theory,<sup>8</sup> Einstein-Gauss-Bonnet gravity,<sup>9</sup> brane worlds<sup>10</sup> and others. Nevertheless, it is highly desirable to further explore such an opportunity in general relativity as a theory with the best experimental status, quite well describing the reality on the macroscopic scale.

Cylindrical wormholes with and without rotation in general relativity were discussed, in particular, in Refs. 11, 12 (see also references therein). It was shown there that phantom-free cylindrical wormhole solutions to the Einstein equations are rather easily obtained, and there are numerous examples of such solutions, but none of them are asymptotically flat.<sup>a</sup> Meanwhile, asymptotic flatness is necessary for considering a wormhole entrance as a local (though extended in one direction) object in our Universe. To overcome this difficulty, we tried in Ref. 12 to build wormhole configurations with flat asymptotic regions on both sides of the throat by matching a wormhole solution to a properly chosen region of flat space-time at some surfaces  $\Sigma_-$  and  $\Sigma_+$ . It was shown, however, that for wormhole solutions with a massless scalar field such a procedure does not solve the problem since one or both thin shells appearing on  $\Sigma_-$  and  $\Sigma_+$  inevitably violate the NEC.

The present paper generalizes Ref. 12 in two respects: (i) we present new exact wormhole solutions with a scalar field as a source of gravity, namely, for a massless scalar in the presence of a cosmological constant and for a self-interacting field with an exponential potential, and (ii) we prove a no-go theorem saying that the above-mentioned cut-and-paste procedure does not lead to asymptotically flat phantom-free wormholes if the source of gravity in the wormhole solution is a minimally coupled scalar field with an arbitrary potential.

<sup>a</sup>Or at least asymptotically flat up to an angular deficit appearing in cosmic string configurations. In what follows, for brevity, we will only mention asymptotic flatness although the same can be said about such cosmic-string asymptotics.

## 2. Basic Equations

A stationary cylindrically symmetric metric with rotation can be written as

$$ds^2 = e^{2\gamma(u)}[dt - E(u)e^{-2\gamma(u)}d\varphi]^2 - e^{2\alpha(u)}du^2 - e^{2\mu(u)}dz^2 - e^{2\beta(u)}d\varphi^2, \quad (1)$$

where the second line gives the three-dimensional line element,  $u$  is any admissible radial coordinate,  $z \in \mathbb{R}$  and  $\varphi \in [0, 2\pi)$  are the longitudinal and angular ones, respectively. There are two reasonable definitions of a cylindrical wormhole throat:<sup>11</sup> (i) as a regular minimum of the circular radius  $r(u) = e^{\beta(u)}$  (to be called an  $r$ -throat) and (ii) as a regular minimum of the area function  $a(u) = e^{\mu+\beta}$  (to be called an  $a$ -throat).

A new feature of (1) as compared to the static cylindrical metric (the same metric with  $E \equiv 0$ ) is the emergence of a vortex gravitational field described as a 4-curl of the tetrad  $e_a^\mu$ : its kinematic characteristic is the angular velocity of tetrad rotation<sup>13</sup>

$$\omega^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}e_{m\nu}e_{\rho;\sigma}^m, \quad (2)$$

where the Latin letters  $m, n, \dots$  stand for Lorentz indices. In our case, with an arbitrary radial coordinate  $u$ , the vortex  $\omega = \sqrt{\omega_\alpha\omega^\alpha}$  is<sup>12-14</sup>

$$\omega = \frac{1}{2}(Ee^{-2\gamma})'e^{\gamma-\beta-\alpha}. \quad (3)$$

Furthermore, the off-diagonal component of the Ricci tensor  $R_0^3$  in the gauge  $\alpha = \mu$  is given by

$$\sqrt{-g}R_0^3 = -(\omega e^{2\gamma+\mu})', \quad g := \det(g_{\mu\nu}). \quad (4)$$

Assuming that our rotating reference frame is comoving to the matter source of gravity, that is, the azimuthal flow  $T_0^3 = 0$ , we find from  $R_0^3 = 0$  that

$$\omega = \omega_0 e^{-\mu-2\gamma}, \quad \omega_0 = \text{const}, \quad (5)$$

and this relation is valid, by construction, in an arbitrary gauge.

As a result,<sup>12</sup> the diagonal components of the Ricci tensor  $R_\mu^\nu$  can be written as the corresponding components  ${}_sR_\mu^\nu$  for the static metric (with  $E = 0$ ) plus the  $\omega$ -dependent addition

$$\omega R_\mu^\nu = \omega^2 \text{diag}(-2, 2, 0, 2), \quad (6)$$

The Einstein tensor  $G_\mu^\nu = R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R$  splits in a similar manner,  $G_\mu^\nu = {}_sG_\mu^\nu + \omega G_\mu^\nu$ , where

$$\omega G_\mu^\nu = \omega^2 \text{diag}(-3, 1, -1, 1). \quad (7)$$

One can check that the tensors  ${}_sG_\mu^\nu$  and  $\omega G_\mu^\nu$  (each separately) satisfy the ‘‘conservation law’’  $\nabla_\alpha G_\mu^\alpha = 0$  with respect to the static metric (with  $E = 0$ ).

Then, according to the Einstein equations  $G_\mu^\nu = -\kappa T_\mu^\nu$ , the tensor  $\omega G_\mu^\nu/\kappa$  behaves as an additional SET with quite exotic properties (thus, the effective energy

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density is  $-3\omega^2/\varkappa < 0$ ), acting in the auxiliary static metric (1) with  $E = 0$ . In its presence, it is rather easy to fulfil the throat existence conditions for both  $r$ - and  $a$ -throats, which is confirmed by some examples.<sup>12,14</sup>

In what follows we will give some new examples of wormhole solutions using as a source of gravity a minimally coupled scalar field  $\phi$  with a self-interaction potential  $V(\phi)$ . Its Lagrangian is

$$L_s = \frac{1}{2}\varepsilon\partial_\alpha\phi\partial^\alpha\phi - V(\phi), \quad (8)$$

where  $\varepsilon = +1$  corresponds to a normal scalar field and  $\varepsilon = -1$  to a phantom one. Let us assume  $\phi = \phi(u)$  and the comoving reference frame in the metric (1), so that  $T_0^3 = 0$ , so that the stress-energy tensor of  $\phi$  is

$$T_\mu^\nu(\phi) = \frac{\varepsilon}{2}e^{-2\alpha}\phi'^2 \text{diag}(1, -1, 1, 1) + \delta_\mu^\nu V(\phi). \quad (9)$$

Then, in the harmonic gauge<sup>15,16</sup>

$$\alpha = \beta + \gamma + \mu, \quad (10)$$

the Einstein-scalar equations take a particularly simple form, and their relevant combinations can be written as follows:

$$\varepsilon e^{-2\alpha}\phi'' = dV/d\phi, \quad (11)$$

$$e^{-2\alpha}\mu'' = \varkappa V, \quad (12)$$

$$\beta'' - \gamma'' = 4\omega_0^2 e^{2\beta-2\gamma}, \quad (13)$$

$$2\mu'' - \beta'' - \gamma'' = 0, \quad (14)$$

$$e^{-2\alpha}(\alpha'' - \alpha'^2 + \beta'^2 + \gamma'^2 + \mu'^2) = 2\omega^2 + \varkappa\varepsilon e^{-2\alpha}\phi'^2 - \varkappa V, \quad (15)$$

where the prime denotes  $d/du$ , and (15) is a first integral of the other four equations. Equations (14) and (13) are easily integrated giving

$$2\mu = \beta + \gamma + au, \quad a = \text{const}, \quad (16)$$

$$e^\eta = \frac{1}{2|\omega_0|s(k,u)}, \quad \eta := \beta - \gamma, \quad k = \text{const}, \quad (17)$$

where two more integration constants have been excluded by choosing a scale along the  $z$  axis and the origin of the  $u$  coordinate, and the function  $s(k, u)$  is defined as

$$s(k, u) = \begin{cases} k^{-1} \sinh ku, & k > 0, \quad u \in \mathbb{R}_+; \\ u, & k = 0, \quad u \in \mathbb{R}_+; \\ k^{-1} \sin ku, & k < 0, \quad 0 < u < \pi/|k|. \end{cases} \quad (18)$$

As a result, the metric functions  $\beta, \gamma, \mu$  are expressed in terms of  $\eta(u)$  given by (17) and  $\alpha(u)$  as follows:

$$2\beta = \eta + \frac{1}{3}(2\alpha - au), \quad (19)$$

$$2\gamma = -\eta + \frac{1}{3}(2\alpha - au), \quad (20)$$

$$2\mu = \frac{2}{3}(\alpha + au), \quad (21)$$

while the remaining two unknowns  $\phi(u)$  and  $\alpha(u)$  obey the equations (11) and

$$e^{-2\alpha}\alpha'' = -3\kappa V. \quad (22)$$

In addition, Eq. (15) leads to their first integral

$$\frac{\varepsilon}{2}\kappa\phi'^2 = \kappa V e^{2\alpha} + \frac{1}{3}\alpha'^2 - \frac{1}{12}a^2 - \frac{1}{4}k^2 \text{sign } k. \quad (23)$$

Lastly, the function  $E(u)$  is easily found by integration from Eq. (3) provided the other metric coefficients are known.

### 3. Examples of Wormhole Solutions

One can make a general observation from the above equations without finding exact solutions: in the case  $k < 0$ , if a solution to Eqs. (11), (22), (23) is finite and regular on the segment  $0 \leq u \leq \pi/|k|$  (or, equivalently, on any other half-wave of the function  $\sin ku$ ), then the whole configuration is a wormhole, with both  $r$ - and  $a$ -throats, and both  $r(u)$  and  $a(u)$  tend to infinity at the extremes  $u = 0$  and  $u = \pi/|k|$ , but these ends are singular due to  $e^\gamma \rightarrow 0$  and  $\omega \rightarrow \infty$ .

We will confirm this observation with examples of exact solutions and also find that there can be wormhole solutions with  $k \geq 0$  as well.

#### 3.1. Vacuum and a massless scalar

To begin with, consider the solution without matter or with a massless scalar field corresponding to the Lagrangian (8) with  $V \equiv 0$ . In this case, Eqs. (11) and (12) read  $\phi'' = 0$  and  $\mu'' = 0$ , which, combined with (16), (17) and (15), immediately lead to the complete solution<sup>12, 14, 17</sup>

$$\begin{aligned} e^{2\beta} &= \frac{e^{2hu}}{2|\omega_0|s(k, u)}, & e^{2\mu} &= e^{-2mu}, \\ e^{2\gamma} &= 2|\omega_0|s(k, u)e^{2hu}, & e^{2\alpha} &= e^{(4h-2m)u}, \\ \omega &= (\text{sign } \omega_0) \frac{e^{mu-2hu}}{2s(k, u)}, & \phi &= Cu \\ E &= e^{2hu}[E_0s(k, u) - s'(k, u)], \end{aligned} \quad (24)$$

where  $\omega_0, E_0, h, k, m, C$  are integration constants obeying the relation

$$k^2 \text{sign } k = 4(h^2 - 2hm) - 2\kappa C^2. \quad (25)$$

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If the scalar charge  $C$  is zero, it is a vacuum solution.

In all branches of the solution,  $r \rightarrow \infty$  and  $e^\gamma \rightarrow 0$  as  $u \rightarrow 0$ . In the same limit, the vortex  $\omega \rightarrow \infty$ , indicating a singularity. At the other end of the  $u$  range, the situation is more diverse:

- $k < 0$ . A wormhole geometry (with both kinds of throats) is described by all such solutions. At both ends,  $e^\beta \rightarrow \infty$  and  $e^\gamma \rightarrow 0$ , while  $e^{\beta+\gamma}$  and  $e^\mu$  remain finite, and  $\omega \sim e^{-2\gamma} \rightarrow \infty$ .
- $k = 0$ . At large  $u$  we have  $e^{2\beta} \sim u^{-1}e^{2hu}$  and  $e^{2\gamma} \sim ue^{2hu}$ , hence we have a wormhole geometry with an  $r$ -throat if  $h > 0$  and with an  $a$ -throat if  $h - m > 0$ . In addition,  $e^\gamma \rightarrow \infty$  at large  $u$ .
- $k > 0$ . At large  $u$ ,  $e^{2\beta} \sim e^{(2h-k)u}$  and  $e^{2\gamma} \sim e^{(2h+k)u}$ , hence a wormhole with an  $r$ -throat exists if  $0 < k < 2h$ ; we also have  $e^\gamma \rightarrow \infty$  at large  $u$ . A wormhole with an  $a$ -throat exists if  $0 < k < 2(h - m)$ .

### 3.2. A massless scalar and a cosmological constant

Consider the Lagrangian (8) with  $V = \Lambda/\varkappa = \text{const}$ . Then Eq. (11) again takes the form  $\phi'' = 0$ , leading, as before, to  $\phi = Cu$ , while (22) is a Liouville equation, like (14), which is easily solved. Let us restrict ourselves to the case of larger interest  $\Lambda > 0$ , then Eq. (22) yields

$$e^\alpha = \frac{b}{\sqrt{3\Lambda} \cosh[b(u - u_0)]}, \quad (26)$$

where  $b > 0$  and  $u_0 \in \mathbb{R}$  are integration constants. According to (19)–(21),

$$\begin{aligned} e^{2\beta} &= \frac{e^{(-au+2\alpha)/3}}{2|\omega_0|s(k, u)}, & e^{2\gamma} &= 2|\omega_0|s(k, u)e^{(-au+2\alpha)/3}, \\ e^{2\mu} &= e^{2(\alpha+au)/3}, & \omega &= \frac{(\text{sign } \omega_0)}{s(k, u)} e^{-\alpha} \end{aligned} \quad (27)$$

where  $e^\alpha$  is given by (26). The function  $E(u)$  is found, as before, from (3). The integration constants  $k, a, b, C$  are connected by the following relation due to (23):

$$k^2 \text{sign } k = \frac{4}{3}b^2 - \frac{1}{3}a^2 - 2\varepsilon\kappa C^2. \quad (28)$$

This solution with  $k < 0$  describes a wormhole configuration according to the previous general observation. As to solutions with  $k \geq 0$ , we can look whether or not  $r(u) \equiv e^\beta$  and  $a(u) \equiv e^{\beta+\mu}$  tend to infinity as  $u \rightarrow \infty$  in order to check whether or not there are  $r$ - or  $a$ -throats, respectively. It is easy to verify that an  $r$ -throat is found if  $3k + a + 2b < 0$ , an  $a$ -throat if  $3k - a + 4b < 0$ , and adding these two conditions, we conclude that a wormhole with both kinds of throats exists only if  $k + b < 0$ , which is impossible since  $k \geq 0$  by assumption and  $b > 0$  according to (26). We conclude that this family of solutions describes wormholes with both kinds of throats only in the case  $k < 0$ .

For  $\Lambda < 0$  the solutions are more diverse since Eq. (22) leads to three branches similar to (17), but these solutions are beyond the scope of this paper.

### 3.3. A scalar field with an exponential potential

Now consider the Lagrangian (8) with the potential

$$V(\phi) = V_0 e^{2\lambda\phi}, \quad V_0 = \text{const} > 0. \quad (29)$$

Two combinations of (11) and (22) are then easily integrable:

$$3\varepsilon\kappa\phi'' + 2\lambda\alpha'' = 0, \quad (30)$$

$$(\alpha + \lambda\phi)'' = (2\varepsilon\lambda^2 - 3\kappa)V_0 e^{2\alpha+2\lambda\phi}. \quad (31)$$

Although solutions can readily be found for any values of the parameters involved, we will restrict ourselves to those with  $\varepsilon = +1$  (a non-phantom scalar) and  $L^2 := V_0(2\lambda^2 - 3\kappa) > 0$ . The latter condition is justified since  $\lambda$  is a length related to the field self-interaction and is likely to be of the order inherent to particle physics, so that  $\lambda \gg l_p = \sqrt{G} = \sqrt{\kappa/(8\pi)}$  (the Planck length).

Thus we find from (30) and (31):

$$3\kappa\phi + 2\lambda\alpha = Cu, \quad e^{\alpha+\lambda\phi} = \frac{1}{Ls(h, u - u_1)}, \quad (32)$$

where  $C, h, u_1$  are integration constants and, as before, we suppress one more constant by choosing the zero point of  $\phi$ ; the function  $s(h, u - u_1)$  is defined by (18) with proper substitutions. The integration constants are connected by the following relation due to (23):

$$2C^2 - 12\kappa h^2 \text{sign } h = (2\lambda^2 - 3\kappa)(a^2 + 3k^2 \text{sign } k). \quad (33)$$

For  $\alpha(u)$  we obtain

$$e^\alpha = e^{-\frac{\lambda Cu}{2\lambda^2 - 3\kappa}} \left[ L s(h, u - u_1) \right]^{-\frac{3\kappa}{2\lambda^2 - 3\kappa}}. \quad (34)$$

The other metric coefficients are now easily found using (19)–(21),  $\phi(u)$  is determined from the first equality in (32), and then  $\omega$  and  $E$  from (5) and (3), respectively.

The solution behavior is rather diverse, depending on the interplay of zeros of the functions  $s(k, u)$  and  $s(h, u - u_1)$ . We will not describe all variants in detail but only notice that there are wormhole solutions with  $k < 0$  according to the general description at the beginning of this section in all cases in which  $s(h, u - u_1) > 0$  on the segment  $0 \leq u \leq \pi/|k|$ . There also exist some wormhole solutions with  $k \geq 0$ , not to be considered here; a full description of this solution is postponed for future work.

## 4. Asymptotic Flatness and Thin Shells: a No-Go Theorem

If we wish to describe wormholes in our weakly curved Universe, potentially visible to distant observers like ourselves, it is necessary to suppose their flat (or string) asymptotic behaviors. However, it is hard to achieve when dealing with cylindrically

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symmetric systems: indeed, even the Levi-Civita vacuum solution is asymptotically flat only in the special case where the space-time is simply flat.

A possible way out is to try to cut a non-asymptotically flat wormhole configuration at some regular cylinders  $u = u_+$  and  $u = u_-$  on different sides of the throat and to match it there to properly chosen flat space regions. The junction surfaces will then comprise thin shells with certain surface densities and pressures, and we should check whether or not they satisfy the standard energy conditions.

To do so, one should take the flat-space metric in a rotating reference frame: in the Minkowski metric  $ds^2 = dt^2 - dx^2 - dz^2 - x^2 d\varphi^2$ , substituting  $\varphi \rightarrow \varphi + \Omega t$ , we obtain

$$ds_{\text{M}}^2 = dx^2 + dz^2 + x^2(d\varphi + \Omega dt)^2 - dt^2, \quad (35)$$

$\Omega = \text{const}$  being the angular velocity of the reference frame. The relevant quantities defined above are, in the notations of (1),

$$g_{00} = e^{2\gamma} = 1 - \Omega^2 x^2, \quad r^2 \equiv e^{2\beta} = \frac{x^2}{1 - \Omega^2 x^2},$$

$$E = \Omega x^2, \quad \omega = \frac{\Omega}{1 - \Omega^2 x^2} \quad (36)$$

This metric is stationary and can be matched with an internal metric at  $|x| < 1/|\Omega|$ , inside the “light cylinder” on which the linear rotational velocity reaches that of light.

Matching of two cylindrically symmetric regions at a surface  $\Sigma : u = u_0$  requires, above all, that this surface be the same as seen from both sides, therefore,

$$[\beta] = 0, \quad [\mu] = 0, \quad [\gamma] = 0, \quad [E] = 0, \quad (37)$$

where, as usual, the square brackets denote discontinuities across the surface in question: for any  $f(u)$ ,  $[f] = f(u_0 + 0) - f(u_0 - 0)$ . One should note that in general the metrics on different sides of  $\Sigma$  may be written using different gauges (choices of the radial coordinate  $u$ ), but it is unimportant since the quantities involved in all matching conditions used are insensitive to the choice of  $u$ .

The next step is to determine the material content of the junction surface  $\Sigma$  according to the Darmois-Israel formalism:<sup>18,19</sup> in our case of a timelike  $\Sigma$ , the surface stress-energy tensor  $S_a^b$  is given by

$$S_a^b = -\frac{1}{\varkappa}[\tilde{K}_a^b], \quad \tilde{K}_a^b := K_a^b - \delta_a^b K, \quad (38)$$

where  $K = K_a^a$ ,  $K_a^b$  is the extrinsic curvature of the surface  $\Sigma$ , and (since  $\Sigma$  is  $x^1 = \text{const}$ ) the indices  $a$  and  $b$  take the values 0, 2, 3.

Let us now assume that the internal region, containing both  $r$ - and  $a$ -throats, is described by a solution to the Einstein-scalar equations corresponding to the Lagrangian (8) with a certain  $V(\phi)$ . The task is to choose the surfaces  $\Sigma_{\pm}$ , to perform matching and to calculate the surface densities and pressures.



The matching conditions (37) on  $\Sigma_{\pm}$ , identifying the surfaces  $x = x_{\pm}$  in Minkowski regions and  $u = u_{\pm}$  in the internal region, are fulfilled by fixing the values of  $x_{\pm}$  for given  $u_{\pm}$ , the scales along the  $t$  and  $z$  axes, and the values of  $\Omega = \Omega_{\pm}$  in each Minkowski region. It is important that we should take  $x_+ > 0$  and  $x_- < 0$  to adjust the directions of the normal vectors to  $\Sigma_{\pm}$ .

Now, the question is whether the surface stress-energy tensors on  $\Sigma_{\pm}$  can satisfy the WEC under some values of the system parameters. A criterion for that is the validity of the WEC which includes the requirements

$$\frac{S_{00}}{g_{00}} = \sigma \geq 0, \quad S_{ab}\xi^a\xi^b \geq 0, \quad (39)$$

where  $\xi^a$  is any null vector ( $\xi^a\xi_a = 0$ ) on  $\Sigma = \Sigma_{\pm}$ , *i.e.*, the second inequality in (39) comprises the NEC as part of the WEC. The conditions (39) are equivalent to

$$[\tilde{K}_{44}/g_{44}] \leq 0, \quad [K_{ab}\xi^a\xi^b] \leq 0. \quad (40)$$

If we choose two null vectors on  $\Sigma$  in the  $z$  and  $\varphi$  directions as

$$\xi_{(1)}^a = (e^{-\gamma}, e^{-\mu}, 0), \quad \xi_{(2)}^a = (e^{-\gamma} + Ee^{-\beta-2\gamma}, 0, e^{-\beta}), \quad (41)$$

the conditions (40) read<sup>12</sup>

$$[e^{-\alpha}(\beta' + \mu')] \leq 0, \quad [e^{-\alpha}(\mu' - \gamma')] \leq 0, \quad [e^{-\alpha}(\beta' - \gamma') + 2\omega] \leq 0. \quad (42)$$

Now we can apply these requirements to our configuration at both junctions. It turns out that for our purposes it is sufficient to use only the third condition which contains the function  $\eta = \beta - \gamma$  given by Eq. (17) for solutions with any  $V(\phi)$ . Using it, on  $\Sigma_-$  with  $x = x_- < 0$  we obtain

$$e^{-\alpha(u_-)} \frac{(-s' + \text{sign } \omega_0)}{s} + \frac{(1 + \Omega_- x)^2}{|x|(1 - \Omega_-^2 x^2)} \leq 0, \quad (43)$$

and on  $\Sigma_+$  with  $x = x_+ > 0$  we have in a similar way

$$e^{-\alpha(u_+)} \frac{(s' - \text{sign } \omega_0)}{s} + \frac{(1 + \Omega_+ x)^2}{x(1 - \Omega_+^2 x^2)} \leq 0. \quad (44)$$

Here  $s$  and  $s' = ds/du$  refer to the function  $s = s(k, u)$  introduced in (18).

The inequalities (43) and (44) lead to the conclusion that the matter content of both  $\Sigma_+$  and  $\Sigma_-$  cannot satisfy the NEC (hence also the WEC).

Indeed, if  $\omega_0 > 0$ , the inequality (43) can only hold if  $1 - s'(k, u) < 0$  at  $u = u_-$ . But  $s'(k, u) = \{\cosh ku, 1, \cos |k|u\}$  for  $k > 0$ ,  $k = 0$  and  $k < 0$ , respectively, and only at  $k > 0$  we have  $1 - s' < 0$ . Thus the NEC for  $\Sigma_-$  definitely requires  $k > 0$  in the solution valid in the internal region. In a similar way, (44) can hold only if  $1 - s'(k, u) > 0$  at  $u = u_+$ , and this is only possible if  $k < 0$ . All this means that whatever particular solution (with fixed parameters including  $k$ ) is taken to describe the internal region, the inequalities (43) and (44) cannot hold simultaneously.

If  $\omega_0 < 0$ , the expression  $-s' - 1$  appearing in (43) is negative, so this inequality can hold; however, in (44) there instead appears  $s' + 1 > 0$ , hence this inequality cannot hold whatever be the parameter  $k$ .

We conclude that the NEC is inevitably violated at least on one of the surfaces  $\Sigma_+$  and  $\Sigma_-$ .

## 5. Conclusion

As shown in Ref. 11, static cylindrically symmetric wormholes with two flat (or string) asymptotics in general relativity can only be obtained with negative matter density at least in a certain part of space. We have seen in the present study that if we add rotation, there emerge quite a number of new phantom-free wormhole solutions but none of them are asymptotically flat. Even if we try to inscribe a rotating cylindrical configuration into flat space by matching it to properly chosen parts of flat space, it turns out that this cannot be done simultaneously on both entrances to a wormhole without violating the NEC, as has been proven for such a large class of material sources of gravity as scalar fields (8) with any potentials  $V(\phi)$  (independently of their sign). So the problem of obtaining phantom-free stationary, potentially observable wormholes in general relativity remains open.

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